Recall, from last time,

$$\delta\tilde{\phi} = \delta\phi - \bar{\phi}' T$$

But how does the metric change?

Recall, $ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$
$\qquad\quad = \tilde{g}_{\alpha\beta}(\tilde{x})\, d\tilde{x}^\alpha d\tilde{x}^\beta$

We can write $\quad d\tilde{x}^\alpha = \left(\dfrac{\partial \tilde{x}^\alpha}{\partial x^\mu}\right) dx^\mu \quad$ and $\quad d\tilde{x}^\beta = \left(\dfrac{\partial \tilde{x}^\beta}{\partial x^\nu}\right) dx^\nu$

Thus,

$$g_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \tilde{g}_{\alpha\beta}(\tilde{x})$$

Defining $\tilde{x}^\mu$ as before: $\tilde{x}^\mu(q) := x^\mu(q) + \xi^\mu(q)$ where $\begin{cases} \xi^0 := T \\ \xi^i := L^i = \partial^i L + \hat{L}_i \end{cases}$

we can write

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\mu} = \left(\begin{array}{c|c} \dfrac{\partial \tilde{\eta}}{\partial \eta}\ {\scriptstyle 1\times1} & \dfrac{\partial \tilde{\eta}}{\partial x^j}\ {\scriptstyle 1\times3} \\ \hline \dfrac{\partial \tilde{x}^i}{\partial \eta}\ {\scriptstyle 3\times1} & \dfrac{\partial \tilde{x}^i}{\partial x^j}\ {\scriptstyle 3\times3} \end{array}\right) = \begin{pmatrix} 1 + T' & \partial_j T \\ L^{i\,\prime} & \delta^i_j + \partial_j L^i \end{pmatrix}$$

We label rows by α and we label columns with μ, indexing from 0

$\alpha = 0 \quad \mu = 0$: $\dfrac{\partial \tilde{x}^0}{\partial x^0} = \dfrac{\partial \tilde{\eta}}{\partial \eta} = \dfrac{\partial}{\partial \eta}(\eta + T) = 1 + T'$

$\alpha = 0 \quad \mu = j$: $\dfrac{\partial \tilde{x}^0}{\partial x^j} = \dfrac{\partial \tilde{\eta}}{\partial x^j} = \partial_j(\eta + T) = \partial_j T \quad$ (Since $j \neq 0$)

$\alpha = i \quad \mu = 0$: $\dfrac{\partial \tilde{x}^i}{\partial x^0} = \dfrac{\partial \tilde{x}^i}{\partial \eta} = \dfrac{\partial}{\partial \eta}(x^i + L^i) = (L^i)'$

$\alpha = i \quad \mu = j$: $\dfrac{\partial \tilde{x}^i}{\partial x^j} = \dfrac{\partial}{\partial x^j}(x^i + \xi^i)$
$\qquad\qquad\qquad\qquad = \delta^i_j + \partial_j L^i$

We now investigate what our $g_{\mu\nu} \to \tilde{g}_{\alpha\beta}$ transformation implies for the metric perturbations in our perturbed FRW metric.

P.T.O

e.g. (Temporal component) $\mu = \nu = 0$

$$g_{00}(x) = \frac{\partial \tilde{x}^\alpha}{\partial \eta} \frac{\partial \tilde{x}^\beta}{\partial \eta} \tilde{g}_{\alpha\beta}(\tilde{x})$$

Suppose that $\alpha = 0$ $\beta = i$, then the term is $\frac{\partial \tilde{x}^0}{\partial \eta} \frac{\partial \tilde{x}^i}{\partial \eta} \tilde{g}_{0i}(\tilde{x})$

$$\hookrightarrow = \frac{\partial \tilde{\eta}}{\partial \eta} \cdot L^{i'} \cdot \tilde{g}_{0i}(\tilde{x})$$

$$\propto 1 \cdot \underbrace{\xi^{i'} \cdot \tilde{B}_i}_{\text{the product leads to a 2nd order perturbation}}$$

$\Rightarrow$ it can be neglected

The process is similar for $\alpha = i$ $\beta = j$ and we conclude with the same argument.
Thus, the only contributing term is $\alpha = 0$ $\beta = 0$

$$\Rightarrow \quad g_{00}(x) = \left(\frac{\partial \tilde{\eta}}{\partial \eta}\right) \tilde{g}_{00}(\tilde{x})$$

Recall $ds^2 = a^2(\eta)\left[-(1+2A)\,d\eta^2 + 2B_i\,dx^i\,d\eta + (\delta_{ij} + 2E_{ij})\,dx^i\,dx^j\right.$
Using this and our coordinate transformation we can proceed as follows:

$$g_{00}(x) = \left(\frac{\partial \tilde{\eta}}{\partial \eta}\right)^2 \tilde{g}_{00}(\tilde{x})$$

$$= (1 + T')^2 \tilde{g}_{00}(\tilde{x})$$

Recall $a^2(\tilde{\eta}) = a^2(\eta + T)$

$$\Rightarrow a^2(\eta)(1+2A) = (1+T')^2 a^2(\tilde{\eta})(1 + 2\tilde{A})$$

$$= (1 + 2T' + \cancel{T'^2})(a(\eta) + a'(\eta)T + \cancel{\cdots})^2(1 + 2\tilde{A})$$

Discarding any products higher than second order,

$$= (1 + 2T')(a^2(\eta) + 2a(\eta)a'(\eta)T)(1 + 2\tilde{A})$$

$$= a^2(\eta)(1 + 2\mathcal{H}T + 2T' + 2\tilde{A} + \cancel{\cdots})$$

$$= a^2(\eta)(1 + 2(\tilde{A} + \mathcal{H}T + T'))$$

Thus $\boxed{\tilde{A} = A - \mathcal{H}T - T'}$

We have the full list of results:

$$\boxed{\begin{array}{lll} A \mapsto A - T' - \mathcal{H}T & & \\ B \mapsto B + T - L' & \hat{B}_i \mapsto \hat{B}_i - \hat{L}_i' & \\ C \mapsto C - \mathcal{H}T - \frac{1}{3}\nabla^2 L & & \\ E \mapsto E - L & \hat{E}_i \mapsto \hat{E}_i - \hat{L}_i & \hat{E}_{ij} \mapsto \hat{E}_{ij} \end{array}}$$

We want to define a special combination of the metric perturbations that do not transform under a change of coordinates.

These are constructed as follows:

$$\left.\begin{array}{ll} \Psi := A + \mathcal{H}(B - E') + (B - E')' & \\ \Phi := -C + \frac{1}{3}\nabla^2 E - \mathcal{H}(B - E') & \hat{\Phi}_i := \hat{B}_i - \hat{E}_i' \\ \hat{E}_{ij} & \end{array}\right\}$$

These are the Bardeen variables

↳ these represent the "real" or "true" spacetime perturbations since they are not removable by a gauge transformation.
