Supervision Week 4

This time: Equivalence classes (MA138)
limits of sequences (MA140) - Assignment!

Def^{n}. Sps X is a set.
Sps E is a relation on X

↓

Def^{n}. A relation from $X \to Y$ consists of:

- a domain, X
- a codomain, Y
- a subset $X \times Y$

xRy

e.g. $P = \{(x,y) \in \mathbb{R}^2 \mid x^2 = y\}$

E is an equivalence relation if it is:

i) reflexive $(\forall x \in X,\ xRx)$

\+ ii) symmetric $(\forall x,y \in X,\ xRy \Rightarrow yRx)$

\+ iii) transitive $(\forall x,y,z \in X,\ xRy + yRz \Rightarrow xRz)$

□

Example. L_1, L_2 lines in $\mathbb{R}^2$.

$L_1 R L_2 \iff L_1$ and L_2 have the same gradient (or are both vertical) OR $L_1 = L_2$.

P.T.O.

Let us show this is:

i) reflexive: Suppose L_1 is given by $y = m_1x + c_1$

Clearly $(m_1x + c_1)\,R\,(m_1x + c_1)$

since $m_1 = m_1$!

ii) symmetric: Suppose, now, L_2 is given by $y = m_2x + c_2$

If $L_1\,R\,L_2$ then $m_1 = m_2$ or $L_1 = L_2$

If $m_2 = m_1$ or $L_2 = L_1$

$\Rightarrow L_2\,R\,L_1$

iii) transitive: Omitted for brevity, but follows similarly.

Thus R is an equivalence relation.

Question: Why could we not just use the condition

"$L_1\,R\,L_2$ if L_1 and L_2 never intersect" ?

to get R as an equiv. relation? classic way of stating parallelism.

(Ans: reflexivity fails)

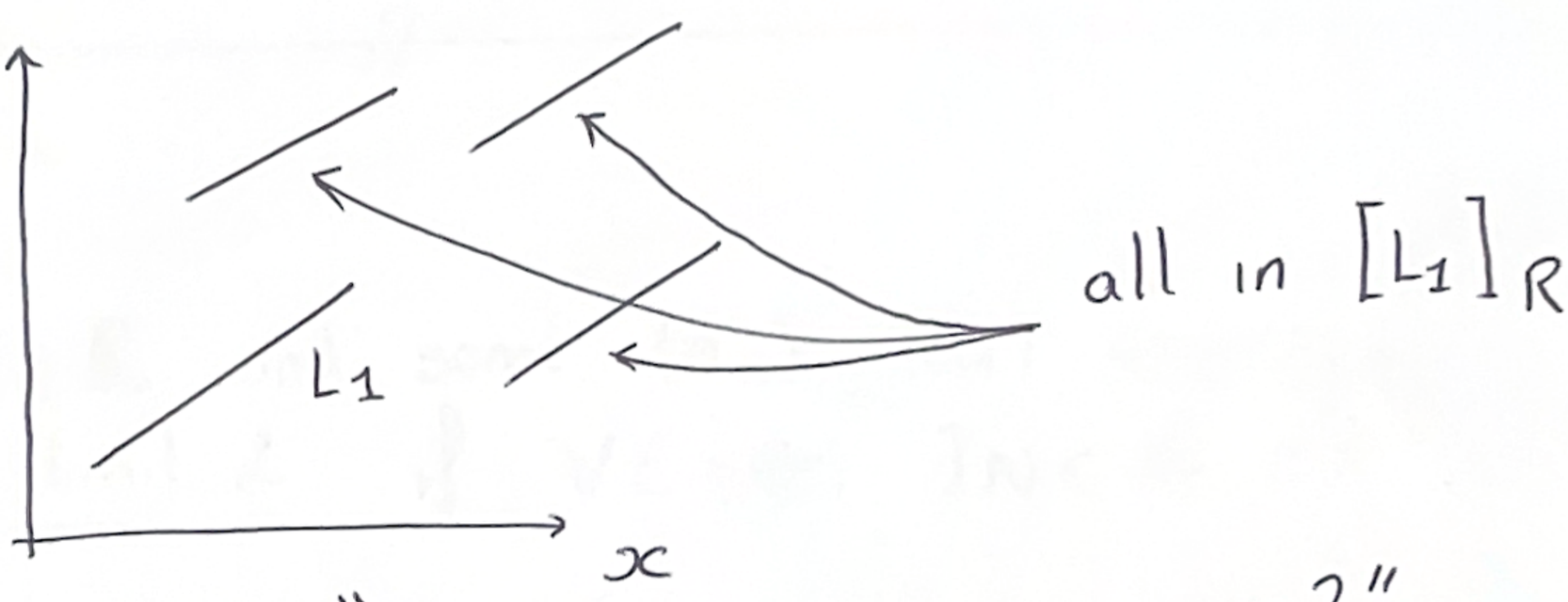
Def^n.

$$[x]_E = \{y \in X \mid xEy\}$$

↑ an equiv. relation on X $(\ni x)$

Question. What does $[L_1]_R$ look like?

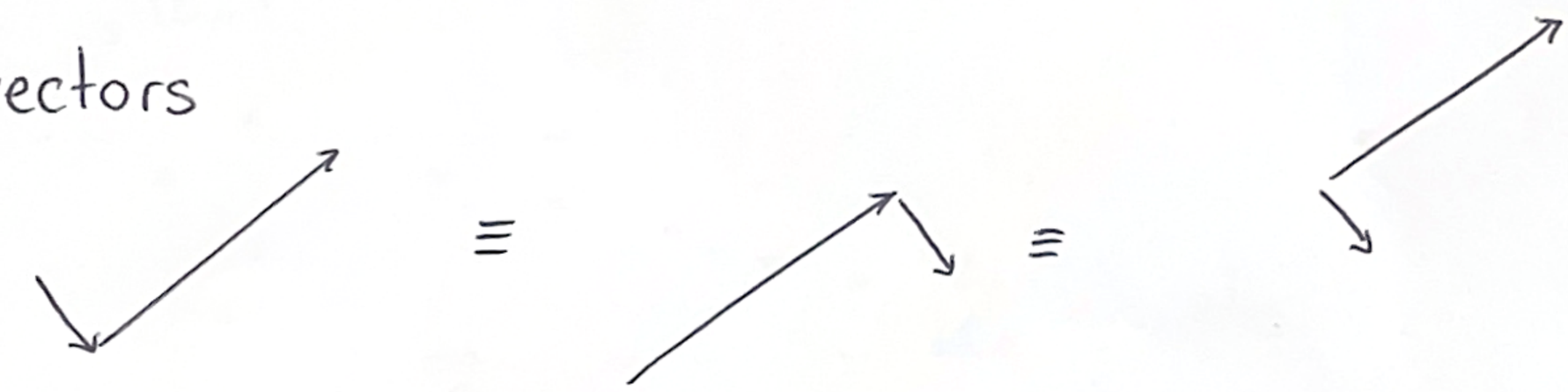
Ans: The set of all lines with the same gradient as L_1.



$\sim$ "$\{mx + c \mid m \bar{\in} m_1, c \in \mathbb{R}\}$"

Why are equivalence classes useful?

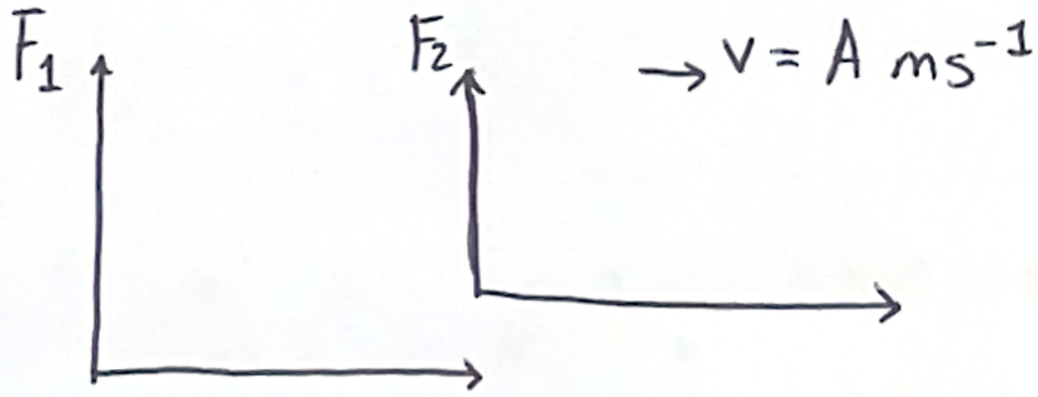
e.g. vectors



e.g. inertial observers (you'll see this in CM + SR)

S the set of all frames

$F_1 \sim F_2$ if they have a constant relative velocity



(can also do this similarly for frames under similar acceleration etc)

The laws of physics are indistinguishible (equivalent) in all F in $\sim$.

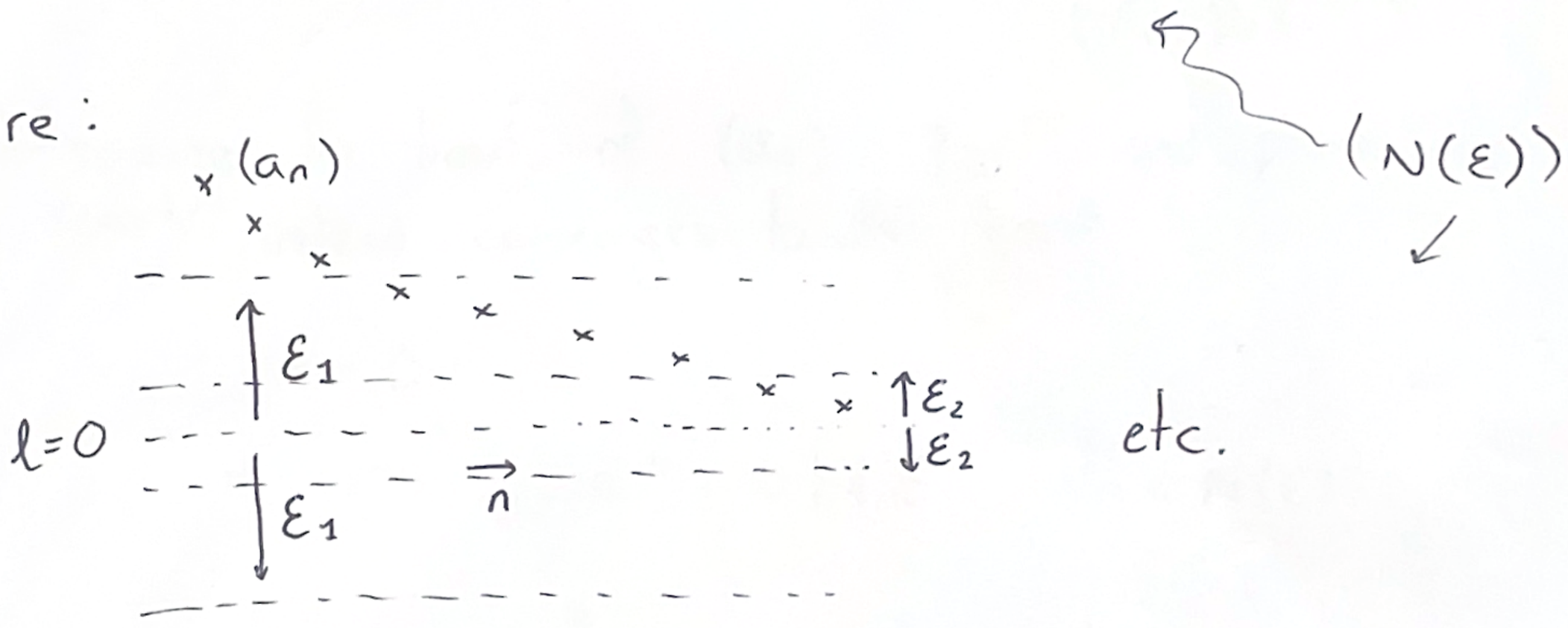
e.g. comoving observers
equivalence of cosmic time ⇝ meaning of age of the universe
⇝ CMB isotropy.

~ Back to MA140 ~

Def^{n}. Given $\ell \in \mathbb{R}$ and some (a_n), (a_n) <u>converges to the limit ℓ</u> if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

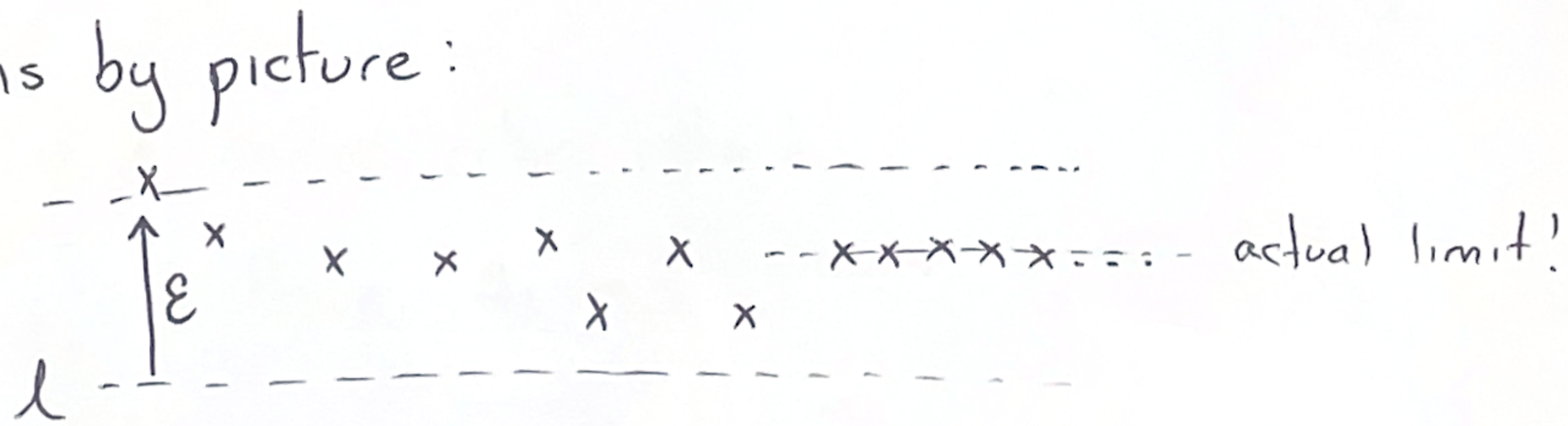
$$|a_n - \ell| < \varepsilon \quad \forall n \geq N$$

Picture:



Question: Why can we not use a definition
"$\exists \varepsilon > 0$ (s.t.) $\exists N \in \mathbb{N}$ s.t. $|a_n - \ell| < \varepsilon$ $\forall n \geq N$"?

Ans by picture:



Questions for supervisees on blackboards:

Q. (ES 2, P3) If $b_n \to 0$ and $|a_n - \ell| \leq |b_n| \ \forall n$ prove that $a_n \to \ell$.

Proof. Pick $\varepsilon > 0$.

By def^n, $\exists N \in \mathbb{N}$ s.t. $|b_n - 0| \leq \varepsilon \ \forall n \geq N$

$(|b_n| \leq \varepsilon)$

$\Rightarrow \forall n \geq N$

$|a_n - \ell| \leq |b_n| \leq \varepsilon$

$\Rightarrow |a_n - \ell| \leq \varepsilon \quad \forall n \geq N$

(The Def^n!)

Q. Determine the limit of $(a_n) = 3e^{-n}$ and prove an (state) indeed converges to the limit.

$\ell = 0$

$\Rightarrow$ WTS. $|3e^{-n} - 0| < \varepsilon \quad \forall n \geq N(\varepsilon)$

$\frac{3}{e^n} < \varepsilon \rightsquigarrow \frac{3}{\varepsilon} < e^n$

holds for

$N > \ln\left(\frac{3}{\varepsilon}\right)$

□

Q (if time, and do before either other).

$(a_n) = 3$. Prove $a_n \to 3$