Supervision Week 5

- Comparison + Ratio test
- (Cauchy sequences) - omitted for now
- Subsequences

But firstly.

Assignment 2:

Q4. lots of marks lost for not being explicit enough in odd/even cases.

Q10. $0 < a \leq b$ gives $\underbrace{b^n < a^n + b^n \leq 2b^n}$ $\forall n \in \mathbb{N}$

setting up the sandwich rule!

n^{th} +ve root $\Rightarrow$ $b \leq (a^n + b^n)^{1/n} \leq \underbrace{2^{1/n}}_{\to 1 \text{ as given in class}} b$

by algebra of limits, $2^{1/n} b \to b$

by ~~sandwich~~ sandwich rule / th^{m}

$(a^n + b^n)^{1/n} \to b$

Question to work on:

Evaluate $\lim_{n\to\infty} (n + b^n)^{\frac{1}{n}}$ where $b \in \mathbb{R}$ $b \geq 1$

Ans: Since $n \geq 1$, $n + b^n > b^n$

Since $b \geq 2$, $b^n > n$ $\forall n \in \mathbb{N}$

Thus we have $n + b^n \leq b^n + b^n = 2b^n$

$\Rightarrow b^n < n + b^n \leq 2b^n$

$\Rightarrow b < (n + b^n)^{1/n} \leq 2^{1/n} b$

$\Rightarrow (n + b^n)^{1/n} \to b$

Lemma. (Comparison test for sequences)

If $(a_n), (b_n)$ seq. s.t. $b_n \to \infty$ as $n \to \infty$

and $\exists N \in \mathbb{N}$ s.t. $a_n \geq b_n$ $\forall n \geq N$

then $a_n \to \infty$, as $n \to \infty$.

Question for board:

Prove $\frac{n^3}{n^2+1} \to \infty$ as $n \to \infty$. $(n \in \mathbb{N})$

Ans: $n^2 \geq 1 \Rightarrow \frac{n^3}{n^2+1} \geq \frac{n^3}{2n^2} = \frac{1}{2} \cdot n$

from class. $n \to \infty$.

□

Lemma. (Ratio test for limits)

(a_n) s.t. $a_n > 0 \quad \forall n \in \mathbb{N}$

Sps $\frac{a_{n+1}}{a_n} \to r \in \mathbb{R}$ as $n \to \infty$ for some $r \in \mathbb{R}$.

If $r < 1$, $a_n \to 0$

If $r > 1$, $a_n \to \infty$

Question. Why don't we assume anything from $r = 1$?

e.g. $a_n = n$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{n+1}{n} = 1 + \frac{1}{n}$$

$(1 + \frac{1}{n}) \to 1$ as $n \to \infty$

$(a_n \to \infty)$!

$a_n = \frac{1}{n}$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{(\frac{1}{n+1})}{(\frac{1}{n})} = \frac{n}{n+1} \overset{*}{\to} 1$$

$(a_n \to 0)$!

$$* \quad \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}} \to 1$$

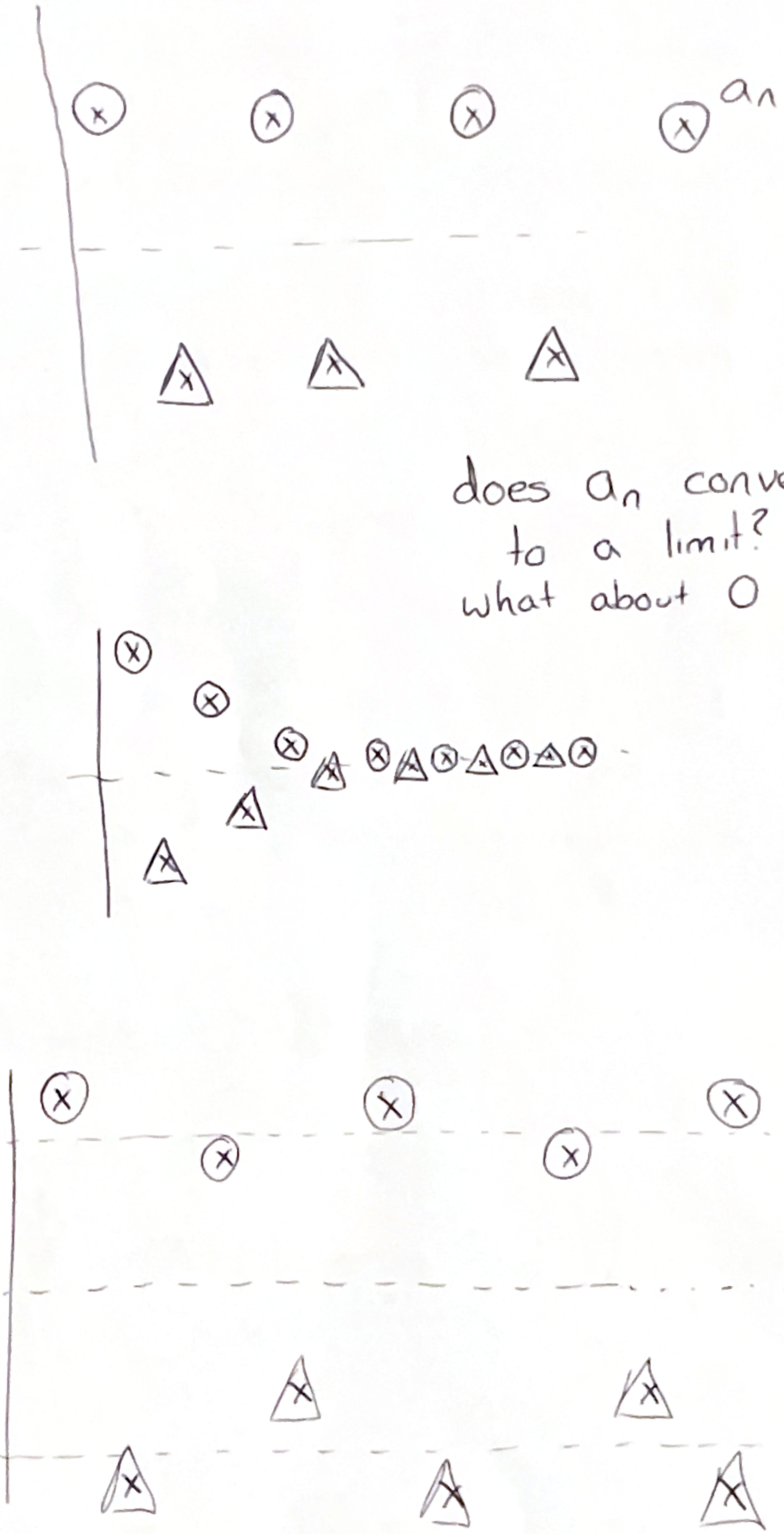
Question for board. Show $a_n = \frac{2^n}{n!}$ converges to 0 as $n \to \infty$.

Ans: $\frac{a_{n+1}}{a_n} = \frac{(\frac{2^{n+1}}{(n+1)!})}{(\frac{2^n}{n!})} = \frac{2n!}{(n+1)!} = \frac{2}{n+1} \to 0$ as $n \to \infty$

Subsequences

I won't give the definition here (you'll see why in a second).

Picture:



Exam-Style Practice.

Q1. Suppose $(a_n) \in \mathbb{R}$ is a sequence indexed by $n \in \mathbb{N}$.

a) What does it mean for the sequence (a_n) to converge to some $l \in \mathbb{R}$ as $n \to \infty$?

b) What do we mean by 'a subsequence of (a_n)'?

c) Give an example of a sequence (a_n) that does not converge but has a convergent subsequence.

d) Suppose (a_n) is given by

$$a_n = (1+n)^{1/n}$$

Show that $a_n \to 1$ as $n \to \infty$.

Note: You may assume that $x^{\frac{1}{n}} \to 1$ as $n \to \infty$ for $x > 0$, and that $n^{1/n} \to 1$ as $n \to \infty$

e) Suppose, instead, that (a_n) is given by

$$a_n = \frac{n}{2^n}$$

Deduce if (a_n) converges and, if so, give its limit as $n \to \infty$.

□