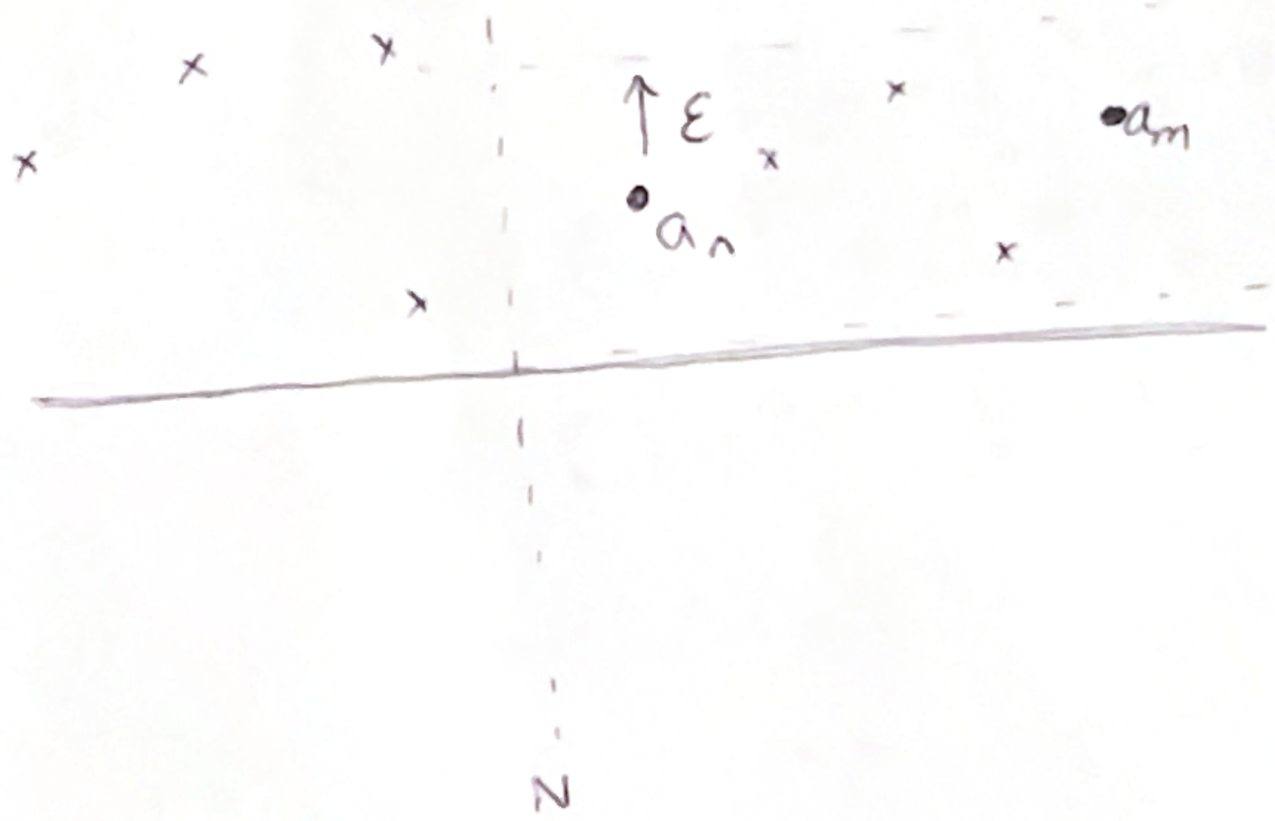
Supervision Wk 6

This time - Cauchy (!) sequences
- Convergence tests for series

Def^n. A sequence (a_n) is Cauchy if $\forall \varepsilon > 0$
$\exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N$
$$|a_n - a_m| < \varepsilon$$



Warm Up: (from lectures)

Prove the lemma:
(a_n) seq, $\ell \in \mathbb{R}$ $a_n \to \ell$ as $n \to \infty$. Then (a_n) Cauchy.

Proof. ... $|a_n - \ell| < \frac{\varepsilon}{2}$
$\forall n, m > \varepsilon$ by triangle inequality.
$$|a_n - a_m| \leq |a_n - \ell| + |\ell - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \square$$

Example. (ES 3, P7)

Sps $|a_{n+1} - a_n| \leq C\alpha^n \quad (n \in \mathbb{N})$

where $C > 0$ and $0 < \alpha < 1$. Prove that (a_n) is Cauchy.

Ans: Pick $\varepsilon > 0$

$|a_m - a_n| =$ ~~$|(a_{m+1} - a_m$~~

(Assume $m > n$) $= |(a_m - a_{m-1}) + (a_{m-1} - a_{m-2})$

$+ \cdots + (a_{n+1} - a_n)|$

By repeated use of triangle inequality

$\leq |a_m - a_{m-1}| + \cdots + |a_{n+1} - a_n|$

$\leq C\alpha^{m-1} + C\alpha^{m-2} + \cdots + C\alpha^n$

$= C(\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n)$

$= C\alpha^n(\alpha^{m-n-1} + \alpha^{m-n-2} + \cdots + 1)$

finite sum of geometric series

$S = 1 + \alpha + \cdots + \alpha^{m-n-2} + \alpha^{m-n-1}$

Since $0 < \alpha < 1$, $S < \sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}$

$\Rightarrow \quad \leq \frac{C\alpha^n}{1-\alpha} \to 0$ from class.

Then $\exists N(\varepsilon)$ s.t.

$$\frac{C\alpha^n}{1-\alpha} < \varepsilon \quad \forall n \geq N$$

Thus for all $m, n \geq N$

$|a_m - a_n| < \varepsilon$ thus (a_n) is Cauchy.

□

If done on the board use these hints:

Hints

1. Get the definition of Cauchy ($|a_m - a_n|$) into a form you can use the T.I. on.
2. Manipulate the resulting expression into a known type of 'sequence'
3. Impose, somehow, an ε inequality
4. Conclude.

Just because we should never forget it:

Thm. (Bolzano - Weierstrass) Any bounded sequence of $\mathbb{R}$ numbers contains a convergent subsequence.

Lets use this to prove:

Let (a_n) be a Cauchy sequence of $\mathbb{R}$-numbers.

Prove (a_n) converges.

Note: Similar to the lecture notes ~ exam style.

Proof. From class,

Lemma 2.42 - Cauchy Sequences are bounded

We know (a_n) is bounded.
Thus we can appeal to the Bolzano-Weierstrass theorem to obtain a subsequence $(a_{n_j})_{j=1}^{\infty}$ and some $\ell \in \mathbb{R}$ s.t. $a_{n_j} \to \ell$ as $j \to \infty$

Now it remains to show $a_n \to \ell$ as $n \to \infty$.

Let $\varepsilon > 0$.

Since $a_{n_j} \to \ell$ as $j \to \infty$ $\exists J \in \mathbb{N}$ s.t.

$$|a_{n_j} - \ell| < \varepsilon/2 \quad \forall j \geq J$$

Since a_n is Cauchy, $\exists N' \in \mathbb{N}$ s.t.

$$|a_n - a_m| < \varepsilon/2 \quad \forall n, m > N'$$

Set $N = \max\{N', n_J\}$.

~~$|a_{n_j} - \ell| =$~~

$$|a_n - \ell| = |a_n - a_{n_j} + a_{n_j} - \ell|$$

$$(\text{triangle ineq.}) \leq \underbrace{|a_n - a_{n_j}|}_{(\text{let } m = n_j)} + \underbrace{|a_{n_j} - \ell|}_{< \varepsilon/2}$$

$$< \varepsilon/2$$

So $|a_n - \ell| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall n > N.$

□

Last time we looked at convergence tests for sequences. Now we want a similar toolbox for series!

Lemma. (Comparison Test for Series)

Suppose: N a positive integer
$(a_n), (b_n)$ are seq. s.t. $0 \leq a_n \leq b_n \quad \forall n \geq N$

Then,

$$\sum_{n=1}^{\infty} b_n < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n < \infty$$

and

$$\sum_{n=1}^{\infty} a_n = \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} b_n = \infty$$

Why is there no $\sum b_n = 0$ case? Ans: Its just trivial.

Example.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

$$\frac{1}{n^2+n} < \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges (from class)}$$

$\uparrow$ $2 > 1$!

Q. What would we have to use if it was $\sum \frac{4}{n^2+n}$ instead?

Q. What if it were $\sum \frac{n}{n^2+n}$

$$= \sum \frac{1}{n+1}, \quad \frac{1}{n+1} \geq \frac{1}{2n}$$

$$\sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n} \to \infty$$