Supervision Wk 7

Examples / Assignment 3

Q1. subsequences
Q3. subsequences
Q10. comparison for $\sum a_n$
Q18. convergence $\underline{\text{iff}}$ (Cauchy condensation)

$\downarrow$

$(\Rightarrow)$ and $(\Leftarrow)$

Subsequences

Def^{n}. Let $(a_n)_{n=1}^{\infty}$ be a sequence.

Let $(n_j)_{j=1}^{\infty}$ be a sequence s.t. $n_j \in \mathbb{N}$ $\forall j \in \mathbb{N}$ and $n_{j+1} > n_j$ $\forall j \in \mathbb{N}$.

This gives a subsequence $(a_{n_j})_{j=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$

Picture as before.

Lemma (from notes) Sps $(a_n)_{n=1}^{\infty}$ a sequence and $\ell \in \mathbb{R}$ satisfies $a_n \to \ell$. Then every subsequence $(a_{n_j})_{j=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$ also satisfies $a_{n_j} \to \ell$ as $j \to \infty$.

P.T.O.

Proof. Let $\varepsilon > 0$.

Since $a_n \to \ell$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$
we have $|a_n - \ell| < \varepsilon$

$(n_j) \geq j \quad \forall j \in \mathbb{N}$. (Why? subsequence of $\mathbb{N}$)

Thus set $J = N$.

If $j \geq J$, $n_j \geq j \geq J = N$

So $|a_{n_j} - \ell| < \varepsilon \quad \forall j \geq J$ □

↑

(Let's try to write this out in two steps:
1. write what we need as the end result
2. write a proof that obtains it.)

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ES 3, Q4.

A convergent sequence has a subsequence converging to ℓ. Show that the original seq. converges to ℓ.

(By what we just proved and uniqueness of limits this follows). Let's prove it properly.

Proof. Let (a_n) be a convergent sequence with $\lim_{n \to \infty} a_n = L \in \mathbb{R}$.

Let (a_{n_k}) be a subsequence s.t. $\lim_{k \to \infty} a_{n_k} = \ell$.

cont'd.

Sps, for contradiction, $L \neq \ell$.

Let us set

$$\varepsilon = \frac{|L - \ell|}{2} > 0$$

By the convergence of a_n, $\exists N_1 \in \mathbb{N}$ s.t.

$$|\overset{a}{x}_n - L| < \varepsilon \quad \forall n \geq N_1$$

By the convergence of a_{n_k}, $\exists K \in \mathbb{N}$ s.t.

$$|a_{n_k} - \ell| < \varepsilon \quad \forall k \geq K$$

Pick $j \geq \max(N_1, k)$.

$$|L - \ell| = |L - \overset{a}{x}_{n_j} + a_{n_j} - \ell|$$

$$\leqslant |L - a_{n_j}| + |a_{n_j} - \ell|$$

$$n_j > N_1$$

$$< \varepsilon + \varepsilon = 2\varepsilon$$

But then $|L - \ell| < |L - \ell|$

which is a contradiction.

□

We now cover what was meant to be done last week.

c.f. Week 6 - page 5