

Dissertation Weekly Progress Report

Week 1

Date of Last Meeting: 1 October 2026
Date of Next Meeting: 8 October 2026
Current Focus Area: Isocurvature Perturbations and Induced Gravitational Waves

1 Introduction

In our last meeting, Siri and I discussed covering isocurvature perturbations and finding links between these and active areas of research in cosmology. Upon first exploring the literature, I found an interesting paper [1] linking these perturbations to gravitational waves (GW). In this summary, I will endeavour to cover a summary of isocurvature perturbations and start to explore the application to GW, and possibly other areas if I find the time.

2 Isocurvature Modes

According to [2], “Isocurvature perturbations represent variations in relative number densities while maintaining homogeneous total energy density and can arise from multifield inflation models, phase transitions, or from Poisson noise in early universe structure formation.”

In this first treatment, we will use a formalism like that presented in [1], differing only by the sign convention in the perturbed metric.

Let us first recall the perturbed metric (in the Newtonian gauge) from Baumann (which is the sign formalism I will, for now, continue to use):

$$ds^2 = a^2(\eta) \left[-(1 + 2\Psi)d\eta^2 + ((1 - 2\Phi)\delta_{ij} + h_{ij})dx^i dx^j \right]$$

In the absence (or neglect of) anisotropic stress we can take $\Psi = \Phi$ giving us

$$ds^2 = a^2(\eta) \left[-(1 + 2\Phi)d\eta^2 + ((1 - 2\Phi)\delta_{ij} + h_{ij})dx^i dx^j \right] \quad (2.1)$$

which is the form which we will use in this discussion.

As in [1], we consider a FLRW universe filled with two components; one behaves like non-relativistic matter whilst the other behaves as a adiabatic, perfect fluid. We endow the latter with an equation of state given by the ratio of the fluid’s pressure to its energy density, $w = P/\rho$ with $0 < w \leq 1$. Thus, we have $\delta\rho = \delta\rho_m + \delta\rho_w$.

Definition 2.1 (*Entropy Perturbation*). In the case we are currently discussing,

$$S := \frac{\delta\rho_m}{\rho_m} - \frac{\delta\rho_w}{(1 + w)\rho_w} \quad (2.2)$$

This is often called the *entropy perturbation*, however I will, for now, often refer to it as ‘isocurvature perturbation’.

The form for S can be more generally be written for two species as

$$S_{A,B} := \frac{\delta_A}{1 + w_A} - \frac{\delta_B}{1 + w_B}$$

where $\delta_A := \delta\rho_A/\bar{\rho}_A$ ¹. It is often useful to choose a species of reference and write the other perturbations relative to this. Let us use the example of choosing the photons as the reference species:

$$\begin{aligned} S_b &\equiv \delta_b - \frac{3}{4}\delta_\gamma, \\ S_c &\equiv \delta_c - \frac{3}{4}\delta_\gamma, \\ S_\nu &\equiv \frac{3}{4}\delta_\nu - \frac{3}{4}\delta_\gamma, \end{aligned}$$

These are the isocurvature modes of their respective species. In the case of the adiabatic mode, $S_b = S_c = S_\nu = 0$.

Definition 2.2 (*Speed of sound in the two fluid system*).

$$c_s^2 = \frac{c_w^2(1+w)\rho_w}{\rho_m + (1+w)\rho_w} \quad (2.3)$$

Derivation. We define the speed of sound to be $c_s^2 := \left(\frac{\partial P}{\partial \rho}\right)_S = \frac{\bar{P}'}{\bar{\rho}'}$. We use (taking $c = 1$) the continuity equation $\dot{\rho} + 3\mathcal{H}(\rho + P) = 0$. Using the fact that the components of the pressure and density sum, and that $P_m = 0$, the result follows trivially. $\square$

Now let us recall the 00- and ij - components of the (linearised) Einstein equation:

$$\begin{aligned} \nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) &= 4\pi G a^2 \delta\rho \\ \Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi &= 4\pi G a^2 \delta P \end{aligned}$$

We can choose units where we set $M_{pl}^2 := \frac{1}{8\pi G} = 1$ which will simplify the notation. Furthermore, let us use that $\delta P = \delta P_m + \delta P_w = \delta P_w = c_w^2 \delta\rho_w$. Combining both of these gives us

$$2\nabla^2\Phi - 6\mathcal{H}(\Phi' + \mathcal{H}\Phi) = a^2(\delta\rho_w + \delta\rho_m) \quad (2.4)$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = \frac{1}{2}a^2 c_w^2 \delta\rho_w \quad (2.5)$$

Using both 2.4 and 2.5 (and using Fourier replacement $\nabla^2 \rightarrow -k^2$) along with definitions 2.2 and 2.3 we can obtain

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' + ((1 + 3c_s^2)\mathcal{H}^2 + 2\mathcal{H}' + c_s^2 k^2)\Phi = -\frac{1}{2}a^2 c_s^2 \rho_m S \quad (2.6)$$

By 2.6, if there is a non-vanishing isocurvature fluctuation $S \neq 0$ at early times, it will source curvature perturbations as the universe evolves. This remains the case even if the initial curvature perturbation vanishes, $\Phi = 0$.

To illustrate this claim, we consider a radiation-dominated universe ($a(\eta) \propto (\eta)$ and $c_s^2 = 1/3$) and impose the initial conditions $\Phi(\eta = 0) = 0$ and $\Phi'(\eta = 0) = 0$, then we compute the evolution governed by 2.6

¹I will often drop this overline notation

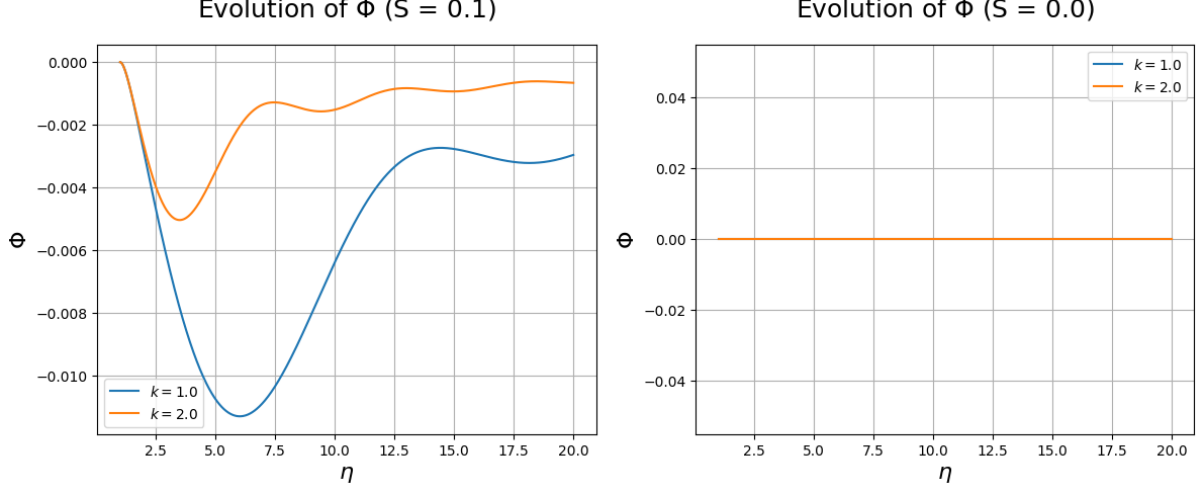


Figure 1: The evolution across conformal time of Φ , in a radiation-dominated universe, with the initial conditions described in the text. The left plot shows the case where there is a constant non-zero isocurvature perturbation S (set arbitrarily to 0.1) whilst the right shows the case with no isocurvature perturbation. We present the results for two values of k as an example.

Since a non-zero isocurvature perturbation sources a curvature perturbation this will, at second order, source tensor perturbations (i.e. GWs). Such signatures could potentially allow us to probe these small-scale fluctuations.

We can derive (though I will not explicitly state the process here) the following equation for the time evolution of S

$$S'' + (1 + 3(c_s^2 - w)) \mathcal{H}S' - k^2(c_s^2 - w)S = -\frac{2c_s^2}{a^2(1+w)\rho_w}k^4\Phi \quad (2.7)$$

Along with 2.6, this yields a closed system for the evolution of Φ and S . This system can be solved analytically ‘deep inside the early, w -dominated epoch’, and is documented in [3, 4].

I will now endeavour to learn the Green’s Function Method which appears to be ubiquitous in the study of induced gravitational waves [5, 6]. This will allow me to study analytical solutions to the $\Phi(\equiv \Phi_{\text{iso}})$ evolution equations, as well as later the motion equations for tensor perturbations and the evolution of S .

3 Green’s Function Method

I am currently working through this using ([7], chp. 7). I predict I will not be able to write a full summary of this in this weeks summary. If we decide to continue on this induced GWs path, I will ensure I include this section at the beginning of next week’s summary.

4 Points for the Next Meeting

In our next meeting, I propose we discuss the following points:

- Is this scalar induced GWs direction something promising enough to continue to explore and evaluate?
- 4 of the 7 references I have used this week are from the same lead author ([1, 5, 3, 4]); is this a particular problem? Are we going too niche, or is this a chance to do something novel?

- How analytically should we treat these equations? Would taking a more numerical approach be more fitting for the dissertation?

For documentation purposes, I will include any concrete answers to these questions and any other major points of dicussion on my website alongside this document.

A Derivation of 2.6

Disclaimer: This section has been transcribed by Gemini AI, based off my hand-written derivation (which can be found in the ‘rough notes’ section of my website). I have fully checked the accuracy of this, but I will of course write this out myself if the derivation is required for any final or interim report.

Step 1: Relation for $a^2 S \rho_\omega$

Starting from the relation for $\delta \rho_\omega$:

$$a^2 c_\omega^2 \delta \rho_\omega = 2 [\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi] \quad (\text{A.1})$$

Rearranging for $a^2 S \rho_\omega$:

$$a^2 S \rho_\omega = \frac{2}{c_\omega^2} [\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi] \quad (\text{A.2})$$

Step 2: Substituting into the Expression for $a^2 S$

Using the definition $a^2 S = \frac{a^2 \delta \rho_m}{\rho_m} - \frac{a^2 \delta \rho_\omega}{(1+\omega)\rho_\omega}$ and substituting $\delta \rho_m$:

$$a^2 S = \frac{-2k^2\Phi - 6\mathcal{H}(\Phi' + \mathcal{H}\Phi) - a^2 S \rho_\omega}{\rho_m} - \frac{a^2 \delta \rho_\omega}{(1+\omega)\rho_\omega} \quad (\text{A.3})$$

Factoring out terms and using the definition of the speed of sound $c_s^2 = \frac{c_\omega^2(1+\omega)\rho_\omega}{\rho_m + (1+\omega)\rho_\omega}$:

$$a^2 S = -\frac{2k^2\Phi + 6\mathcal{H}(\Phi' + \mathcal{H}\Phi)}{\rho_m} - \left(\frac{1}{\rho_m} + \frac{1}{(1+\omega)\rho_\omega} \right) \left(\frac{2}{c_\omega^2} [\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi] \right) \quad (\text{A.4})$$

Using $\frac{1}{\rho_m c_s^2} = \frac{\rho_m + (1+\omega)\rho_\omega}{\rho_m c_\omega^2 (1+\omega)\rho_\omega}$, this simplifies to:

$$a^2 S = -\frac{2k^2\Phi + 6\mathcal{H}(\Phi' + \mathcal{H}\Phi)}{\rho_m} - \frac{2}{\rho_m c_s^2} [\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi] \quad (\text{A.5})$$

Step 3: Rearranging the Differential Equation

Multiplying through by $\frac{1}{2}a^2 c_s^2 \rho_m$:

$$\frac{1}{2}a^2 c_s^2 \rho_m S = (-k^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi)) c_s^2 - \Phi'' - 3\mathcal{H}\Phi' - (2\mathcal{H}' + \mathcal{H}^2)\Phi \quad (\text{A.6})$$

Moving all Φ terms to the left-hand side:

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' + [(1 + 3c_s^2)\mathcal{H}^2 + 2\mathcal{H}' + c_s^2 k^2] \Phi = -\frac{1}{2}a^2 c_s^2 \rho_m S \quad (\text{A.7})$$

References

- [1] Guillem Domènech and Jan Tränkle. “Gravitational waves induced by matter isocurvature in general cosmologies”. In: *Journal of Cosmology and Astroparticle Physics* 2025.12 (Dec. 2025), p. 017. ISSN: 1475-7516. DOI: [10.1088/1475-7516/2025/12/017](https://doi.org/10.1088/1475-7516/2025/12/017). URL: <https://doi.org/10.1088/1475-7516/2025/12/017> (visited on 10/05/2026).
- [2] Chang Han et al. “Infrared behavior of induced gravitational waves from isocurvature perturbations”. In: *Physical Review D* 111.12 (June 26, 2025), p. 123553. DOI: [10.1103/stlv-17yq](https://link.aps.org/doi/10.1103/stlv-17yq). URL: <https://link.aps.org/doi/10.1103/stlv-17yq> (visited on 10/06/2026).
- [3] Guillem Domènech and Jan Tränkle. “From formation to evaporation: Induced gravitational wave probes of the primordial black hole reheating scenario”. In: *Physical Review D* 111.6 (Mar. 13, 2025), p. 063528. ISSN: 2470-0010, 2470-0029. DOI: [10.1103/PhysRevD.111.063528](https://doi.org/10.1103/PhysRevD.111.063528). arXiv: [2409.12125\[gr-qc\]](https://arxiv.org/abs/2409.12125). URL: <http://arxiv.org/abs/2409.12125> (visited on 10/06/2026).
- [4] Guillem Domènech, Samuel Passaglia, and Sébastien Renaux-Petel. “Gravitational waves from dark matter isocurvature”. In: *Journal of Cosmology and Astroparticle Physics* 2022.3 (Mar. 1, 2022), p. 023. ISSN: 1475-7516. DOI: [10.1088/1475-7516/2022/03/023](https://doi.org/10.1088/1475-7516/2022/03/023). arXiv: [2112.10163\[astro-ph.CO\]](https://arxiv.org/abs/2112.10163). URL: <http://arxiv.org/abs/2112.10163> (visited on 10/06/2026).
- [5] Guillem Domènech. “Scalar induced gravitational waves review”. In: *Universe* 7.11 (Oct. 21, 2021), p. 398. ISSN: 2218-1997. DOI: [10.3390/universe7110398](https://doi.org/10.3390/universe7110398). arXiv: [2109.01398\[gr-qc\]](https://arxiv.org/abs/2109.01398). URL: <http://arxiv.org/abs/2109.01398> (visited on 10/07/2026).
- [6] A. J. Iovino et al. *Understanding the Nature of Scalar-Induced Gravitational Waves*. Sept. 29, 2025. DOI: [10.48550/arXiv.2509.24774](https://doi.org/10.48550/arXiv.2509.24774). arXiv: [2509.24774\[gr-qc\]](https://arxiv.org/abs/2509.24774). URL: <http://arxiv.org/abs/2509.24774> (visited on 10/07/2026).
- [7] Philip M. Morse and Herman Feshbach. *Methods of Theoretical Physics, Part I*. 2 vols. New York: McGraw-Hill Book Company, 1953. ISBN: 978-0070433168.