Dissertation Weekly Progress Report
Week 4

Date of Last Meeting: 22 October 2026
Date of Next Meeting: 29 October 2026
Current Focus Area: From Bunch-Davies to Primordial Power

The work this week will mostly assume the reader has a working understanding of quantum mechanics (especially the case of the simple harmonic oscillator in both the Schrodinger and Heisenberg picture). I will, to the extent deemed necessary, provide a full review of the background in-time for the interim report. As usual, the main reference is [**baumann**].

1 Inflationary Vacuum Fluctuations

At first we will work with the case where the background cosmology is taken to be de Sitter space, working with the slow-roll approximation. Thus we have that H is constant and $a = -(H\eta)^{-1}$.

Recall the *Mukhanov-Sasaki* equation:

$$f'' + \left(k^2 - \frac{z''}{z}\right) f = 0 \tag{1.1}$$

Under the assumptions above, that is the slow-roll limit, we have

$$\begin{aligned} \frac{z''}{z} = \frac{a''}{a} \implies \frac{z''}{z} &= \frac{\left(-(H\eta)^{-1}\right)''}{-(H\eta)^{-1}} \\ &= \frac{-2(H\eta^3)^{-1}}{-(H\eta)^{-1}} \\ &= \frac{2}{\eta^2} \end{aligned}$$

which then gives the form of 1.1 as follows

$$f_{\mathbf{k}}'' + \left(k^2 - \frac{2}{\eta^2}\right) f_{\mathbf{k}} = 0 \tag{1.2}$$

This is a good approximation in the spatially flat gauge, before horizon crossing. (There is more story to be said here, but omitted for now)

We now must quantize the field fluctuations. We promote the field to a quantum operator $\hat{f}(\eta, \vec{x})$ along with its conjugate momentum [1] $\pi = f' \to \hat{\pi}(\eta, \vec{x})$. Furthermore, we impose the commutator condition

$$\left[\hat{f}(\eta, \vec{x}), \hat{\pi}(\eta, \vec{x}')\right] = i\delta(\vec{x} - \vec{x}')$$

.

If one Fourier transforms this, we arrive at

$$\left[\hat{f}_{\mathbf{k}}(\eta), \hat{\pi}_{\mathbf{k}}(\eta)\right] = i(2\pi)^3 \delta(\vec{k} + \vec{k}') \tag{1.3}$$

Using the Heisenberg picture, we define a single non-Hermitian operator $\hat{a}_{\mathbf{k}}$. This is motivated by the fact that the operator solution $\hat{f}_{\mathbf{k}}$ is determined by two initial conditions: $\hat{f}_{\mathbf{k}}(\eta_i)$ and $\hat{\pi}_{\mathbf{k}}(\eta_i)$. Furthermore, since the evolution equation 1.2 is linear, the solution will be linear in these operators. With $\hat{a}_{\mathbf{k}}$ such that,

[1] I will aim to include this in an appendix for this week

$$\hat{f}_{\mathbf{k}}(\eta) = f_k(\eta)\hat{a}_{\mathbf{k}} + f_k^*(\eta)\hat{a}^\dagger_{-\mathbf{k}}. \tag{1.4}$$

We impose the cannonical field theory commuation relation for the creation and annhilation operators $(\hat{a}, \hat{a}^\dagger)$

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}\right] = (2\pi)^3\delta(\vec{k} - \vec{k}') \tag{1.5}$$

which gives us a necessary normalisation condition of

$$f_k f_k'^* - f_k' f_k^* = i \tag{1.6}$$

One can now define the vacuum state for this system:

$$\hat{a}_{\mathbf{k}}(\eta_i)\,|0\rangle = 0 \tag{1.7}$$

This encodes the fact that there are no particles that can be annhilated or, more precisely, we cannot remove a particle from this ground state since there are no particles to start with.

Since we are working in an expanding space, the creation and annhilation operators are time dependent so we must define at what time the vacuum state is defined. We do this by taking the limit $\eta_i \to -\infty$.

Recall that the equation of motion (from the Mukhanov-Sasaki equation) is representative of a harmonic oscillator with time-dependent frequency. In the limit given above, we have that $\omega_k \to k$. This is indicative of a simple harmonic oscillator. Recall that the ground state solution to a SHO is

$$q(t) = \sqrt{\frac{\hbar}{2\omega}}e^{-i\omega t}$$

Making the necessary modifications, we can use this to form an initial condition for each Fourier mode:

$$\lim_{k\eta\to-\infty} f_k(\eta) = \frac{1}{\sqrt{2k}}e^{-ik\eta} \tag{1.8}$$

where we have chosen the convention of using natural units $\hbar = 1$.

This boundary condition gives us what we need to solve the Mukhanov-Sasaki equation 1.2. This results in a unique mode function called the *Bunch-Davies mode function*:

$$\boxed{f_k(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}} \tag{1.9}$$

which now gives an 'easy' analytical method to illustrate our idea of the superhorizon freeze-out.

Let us first recall that $\mathcal{R} = f/z$ where $z = (a\bar{\phi}'/\mathcal{H})$. In the approximative regime we are working in we can express z as:

$$z = \frac{\bar{\phi}'}{\mathcal{H}}\frac{-1}{H\eta} = C\frac{-1}{\eta} = A\frac{-1}{k\eta}$$

where C is used to represent the fact that $\bar{\phi}'/\mathcal{H}$ is constant in the limit we are working with, and $A = Ck$ is in turn a constant (since k is a fixed choice).

Using this we can express 1.9 as follows:

$$
\begin{aligned}
\mathcal{R}_k(\eta) &= \frac{1}{z\sqrt{2k}}\left(1-\frac{i}{k\eta}\right)e^{-ik\eta} \\
&= \frac{1}{z\sqrt{2k}}\left(1-\frac{i}{k\eta}\right)\left[\cos(k\eta)-i\sin(k\eta)\right] \\
&= -\frac{k\eta}{A\sqrt{2k}}\left[\left(\cos(k\eta)-\frac{1}{k\eta}\sin(k\eta)\right)-i\left(\sin(k\eta)+\frac{1}{k\eta}\cos(k\eta)\right)\right] \\
&= B\left[-\left(k\eta\cos(k\eta)-\sin(k\eta)\right)+i\left(k\eta\sin(k\eta)+\cos(k\eta)\right)\right]
\end{aligned}
$$

where we have defined $B \coloneqq 1/(A\sqrt{2k})$. Thus we have the results:

$$\mathrm{Re}[\mathcal{R}_k] = B\left[\sin(k\eta) - k\eta\cos(k\eta)\right] \tag{1.10}$$
$$\mathrm{Im}[\mathcal{R}_k] = B\left[k\eta\sin(k\eta) + \cos(k\eta)\right] \tag{1.11}$$

with $|\mathcal{R}_k| = B\sqrt{1+(k\eta)^2}$. Thus it is clear that the 'freezeout value' is determined by the constant B. We can now use these to plot the evolution:

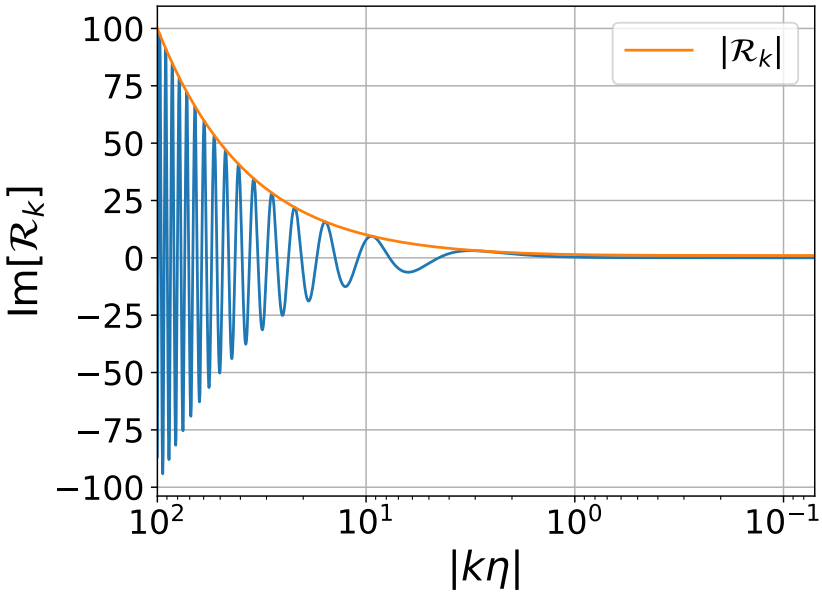


Figure 1: Evolution of the imaginary component of the comoving curvature perturbation using for illustrative reasons $B = 1$ and $k = 1$, As $\eta \to 0_-$ (i.e. as the superhorizon limit is approached) we see the expected freezeout of the perturbation to a constant value.

On superhorizon scales, 1.10 and 1.11 give:

$$\mathrm{Re}[\mathcal{R}_k] = 0$$
$$\mathrm{Im}[\mathcal{R}_k] = B$$

which is easier to visualise if we also take the log of the 'y'-axis...

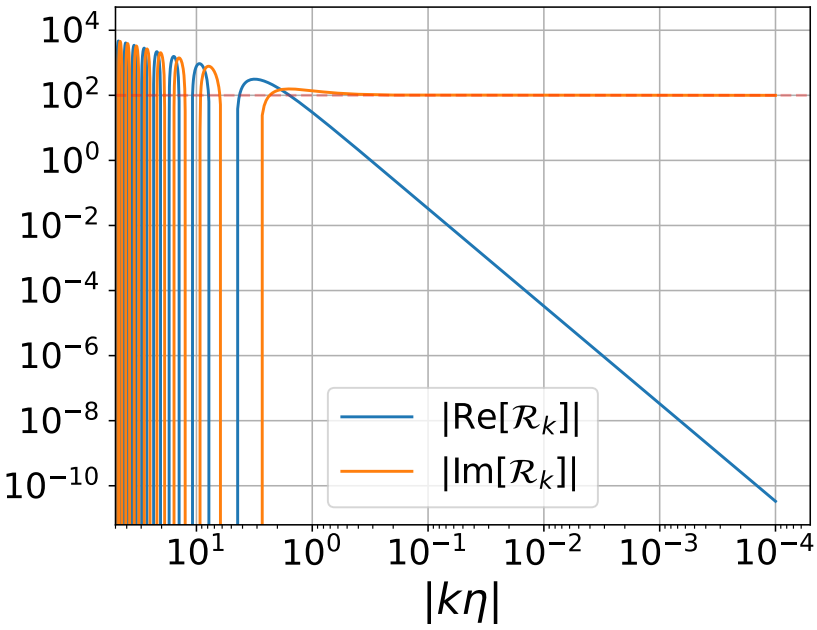


Figure 2: The Re-component of the comoving curvature perturbation tends to zero in the superhorizon limit, leaving only the Im-component which tends to the value of the constant B, signified by the dashed line. Here, we have arbitrarily set $B = 100$.

We have chose to only deal with f_k so far (that is, the fourier transform of f), but now we would like to inspect the *zero-point fluctuations* of $\hat{f}$.

We obtain the operator $\hat{f}$ as follows:

$$\begin{aligned}\hat{f}(\eta, \vec{x}) &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \hat{f}_{\mathbf{k}}(\eta) e^{i\vec{k}\cdot\vec{x}} \\ &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[\hat{a}_{\mathbf{k}} f_k(\eta) e^{i\vec{k}\cdot\vec{x}} + \hat{a}^\dagger_{\mathbf{k}} f^*_k(\eta) e^{-i\vec{k}\cdot\vec{x}}\right]\end{aligned}$$

For completeness, let us first compute $\langle\hat{f}\rangle$. Recall that we define $\hat{a}_{\mathbf{k}}(\eta_i)\,|0\rangle = 0$ and thus $\langle 0|\,\hat{a}^\dagger_{\mathbf{k}}(\eta_i) = 0$. Thus:

$$\begin{aligned}\langle\hat{f}\rangle = \langle 0|\,\hat{f}(\eta, \vec{x})\,|0\rangle &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[f_k(\eta) e^{i\vec{k}\cdot\vec{x}} \underbrace{\langle 0|\,\hat{a}_{\mathbf{k}}\,|0\rangle}_{=0} + f^*_k(\eta) e^{-i\vec{k}\cdot\vec{x}} \underbrace{\langle 0|\,\hat{a}^\dagger_{\mathbf{k}}\,|0\rangle}_{=0}\right] \\ &= 0\end{aligned}$$

We now want to check if the expectation of the square of this operator recerives any zero-point fluctuations. We expect that it does (indeed, it *must*) by the Heinsenberg Uncertainty Principle. Let us do this computation explicitly:

$$
\begin{aligned}
\langle|\hat{f}|^2\rangle &\equiv \langle 0|\hat{f}(\eta,\vec{0})\hat{f}(\eta,\vec{0})|0\rangle \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\int \frac{\mathrm{d}^3k'}{(2\pi)^3}\langle 0|(f_k^*(\eta)\hat{a}^\dagger_{-\mathbf{k}} + f_k(\eta)\hat{a}_{\mathbf{k}})(f_{k'}(\eta)\hat{a}_{\mathbf{k}'} + f^*_{k'}(\eta)\hat{a}^\dagger_{-\mathbf{k}'})|0\rangle \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\int \frac{\mathrm{d}^3k'}{(2\pi)^3} f_k(\eta)f^*_{k'}(\eta)\langle 0|\hat{a}_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}'}|0\rangle \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\int \frac{\mathrm{d}^3k'}{(2\pi)^3} f_k(\eta)f^*_{k'}(\eta)\langle 0|\left([\hat{a}_{\mathbf{k}},\hat{a}^\dagger_{-\mathbf{k}'}] + \hat{a}^\dagger_{-\mathbf{k}'}\hat{a}_{\mathbf{k}}\right)|0\rangle \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\int \frac{\mathrm{d}^3k'}{(2\pi)^3} f_k(\eta)f^*_{k'}(\eta)\langle 0|[\hat{a}_{\mathbf{k}},\hat{a}^\dagger_{-\mathbf{k}'}]|0\rangle \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\int \frac{\mathrm{d}^3k'}{(2\pi)^3} f_k(\eta)f^*_{k'}(\eta)\,(2\pi)^3\delta(\vec{k}+\vec{k}')\langle 0|0\rangle \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}|f_k(\eta)|^2
\end{aligned}
\tag{1.12}
$$

where we have used $\langle 0|0\rangle = 1$ by the normalisation condition, and the canonical commutation relation 1.5.

If we define a dimensionless power spectrum Δ^2_S for a solution S to be given by an integral of the form

$$\int \mathrm{d}\,\ln\,k\;\Delta^2_S(k,\eta)$$

we can transform to spherical coordinates (assuming spatial isotropy) and perform a substituion on 1.12 to arrive at the form

$$\langle|\hat{f}|^2\rangle = \int \mathrm{d}\,\ln\,k\;\frac{k^3}{2\pi^2}|f_k(\eta)|^2$$

Thus we note that the variance is determined by a dimensionless power spectrum

$$\Delta^2_f(k,\eta) = \frac{k^3}{2\pi^2}|f_k(\eta)|^2 \tag{1.13}$$

If we then substitute in 1.9, recalling $f = a\delta\phi$ we arrive at

$$\Delta^2_{\delta\phi}(k,\eta) = \frac{\Delta^2_f(k,\eta)}{a^2(\eta)} = \left(\frac{H}{2\pi}\right)^2\left[1+(k\eta)^2\right] \xrightarrow{k\eta\to 0} \left(\frac{H}{2\pi}\right)^2. \tag{1.14}$$

Recall that we work in the approximation where H is approximately constant, and thus the superhorizon limit of $\Delta^2_{\delta\phi}$ is a constant, notably independent of k. This is the characteristic of a *scale-invariant spectrum*.

This result is, however, not quite physical. Although the Hubble parameter is *approximately* constant, it must remain to some level time-dependent. If we had $\dot{H} = 0$ we wouldn't even have well defined curvature perturbations $\mathcal{R} \propto \dot{H}^{-1}$. Thus we must endow H in 1.14 with some time dependence.

We do this by evaluating 1.14 at the point of horizon crossing $k = aH$. Different fourier modes will thus require a different time at horizon crossing, which gives H a necessary time dependence.

$$\Delta^2_{\delta\phi}(k) \approx \left(\frac{H(t)}{2\pi}\right)^2\Bigg|_{k=aH(t)} \tag{1.15}$$

Recalling

$$\mathcal{R} = \frac{\mathcal{H}}{\bar{\dot{\phi}}'}\delta\phi$$

we can then write the power spectrum of $\mathcal{R}$ as

$$\Delta^2_{\mathcal{R}} = \left(\frac{H}{\bar{\dot{\phi}}}\right)^2 \Delta_{\delta\phi}$$

. If we use the result 1.15 we can write

$$\begin{aligned}\Delta^2_{\mathcal{R}}(k) &= \left(\frac{H^2}{2\pi\dot{\phi}^2}\right)^2\Bigg|_{k=aH} \\ &= \frac{1}{8\pi^2\varepsilon}\frac{H^2}{M^2_{\mathrm{Pl}}}\Bigg|_{k=aH}\end{aligned}$$

We will eventually arrive at the form

$$\Delta^2_R(k) = A_s\left(\frac{k}{k_*}\right)^{n_s-1}$$

. This derivation is omitted for brevity, but I will produce a separate document for this, and eventually will add it as an appendix to this weekly report [2] . I aim to include the derivation of the result to second-order, as seen in **[Stewart_1993]**.

2 Project Progress

This weekly report has been completed in time for the first progress meeting with a member of the Warwick Physics faculty. Any points raised will be added after said meeting.

For the sake of record, I would like to make a small list of goals I want to reach over the next few weeks. I feel it is important that this project sits (as much as possible) at the intersection of theory (that being Inflation) and observation (specifically, the late time structure of the universe; galaxies etc).

Theory

- Summarise the reasons for the characteristic shape of the matter power spectrum
- Begin to look more at the theory of structure formation, although not too much as this is implcitly handled by the computational side. I do not want to spend too much time on 'astrophysics' rather than 'cosmology'!
- How (and why?) do we go from a 3D space power spectrum to an angular power spectrum? / What is the theory behind the limber approximation?
- How do different inflaton potentials effect the $\mathcal{R}$ at freezeout?

Observation + Computation

- Retrieve CMB power spectrum data and implement it in CAMB to improve the constraints we got in Week 3
- How can we best constrain an inflaton potential that has multiple unknowns?
- What signatures would such exotic potentials show, and in what data?

[2] I sadly ran out of time when writing up this report, let this be a friendly warning to myself and any other masters students :)