Dissertation Weekly Progress Report
Week 6+7

Date of Last Meeting: 12 November 2026
Date of Next Meeting: 19 November 2026
Current Focus Area: Primordial Power Spectra

Given I was preparing for an interview last week[1], I did not commit the usual amount of time and thus there was no written report from Week 6 of term. I will summarise the work that still took place alongside the Week 7 content in this report.

1 Numerically Approximating the Primordial Power Spectrum (PPS)

As we progress in the project, it will become useful to have a easy tool for visualising varying inflaton potentials. It would be even better to be able to continue this to a computed *approximation*[2] PPS which could then be passed into CAMB to develop predictions of late-time signatures, *c.f.* previous reports.

We want, preferably, to make no slow-roll approximations. Admittedly, most of our models will be sufficiently slow-roll that we could make-do with the approximative regime, but for the sake of robustness let us work with no approximations (other than of first-order perturbations).

1.0.1 Equations

Although we aim to not use any slow-roll approximations, we will make use of the *slow-roll parameters*.

Recall that we define these as

$$\varepsilon = -\frac{\dot{H}}{H^2} = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} \tag{1.1}$$

$$\delta = \frac{\ddot{\bar{\phi}}}{H\dot{\bar{\phi}}} = 1 - \frac{\bar{\phi}''}{\mathcal{H}\bar{\phi}'}. \tag{1.2}$$

To arrive at a prediction of the PPS, we must collect a set of equations that can work to solve the Mukhanov-Sasaki equation:

$$f_k'' + \left(k^2 - \frac{z''}{z}\right) f_k = 0 \tag{1.3}$$

We have the following relations:

$$-\frac{a^2 V_{,\phi\phi}}{\mathcal{H}^2} = (\delta - 3)(\varepsilon + \delta) - \frac{\delta'}{\mathcal{H}}, \tag{1.4}$$

$$\frac{z''}{z} = \left[2 + 2\varepsilon - 3\delta + (2\varepsilon - \delta)(\varepsilon - \delta) - \frac{\delta'}{\mathcal{H}}\right] \mathcal{H}^2. \tag{1.5}$$

Combining these, we get the following form for the z''/z quotient:

[1] I am pleased to say I got offered the PhD position!

[2] Approximation is used in the sense that we are calculating the value of the PPS at a discrete set of K and then, if required, providing a functional best fit.

$$\frac{z''}{z} = \left[2 + 5\varepsilon + 2\varepsilon^2 - 4\varepsilon\delta\right]\mathcal{H}^2 - a^2 V_{,\phi\phi} \tag{1.6}$$

which leaves us to have expressions for $\mathcal{H}$, $\bar{\phi}'$, and $\bar{\phi}''$. The evolution of a is governed by the equality $a' = a\mathcal{H}$.

To form a coupled ODE solver in Python it suffices to provide an equation for $\bar{\phi}''$ in terms of $\bar{\phi}'$. This can be obtained directly, from the Klein-Gordon equation for the background giving us:

$$\bar{\phi}'' = -2\mathcal{H}\bar{\phi}' - a^2 V_{,\phi}. \tag{1.7}$$

We assume spatial flatness, and that the Universe's energy density is completely dominated by the scalar field. A single light scalar field has energy density $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) = \frac{1}{2a}\phi' + V(\phi)$. Using these assumptions we use the Friedmann equation to obtain an equation for $\mathcal{H}^2$:

$$\begin{aligned} H^2 = \frac{8\pi G}{3}\rho \implies \mathcal{H}^2 &= \frac{8\pi G a^2}{3}\rho \\ &= \frac{8\pi G a^2}{3}\left(\frac{1}{2a^2}\phi'^2 + V(\phi)\right) \\ &= \frac{1}{3M_{\text{Pl}}^2}\left(\frac{1}{2}\phi'^2 + a^2 V(\phi)\right) \end{aligned} \tag{1.8}$$

This gives us all we need to numerically solve 1.3.

To then approximate the PPS we calculate it at a user-designated number of k-values by the equation:

$$P_{\mathcal{R}}(k) \coloneqq \Delta^2_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}\frac{1}{z^2(\eta)}|f_k(\eta)|^2 \tag{1.9}$$

where we define $P_{\mathcal{R}}$ to represent the dimensionless power spectrum.

1.0.2 Usability

We want to be able to quickly depoly this code when testing a new inflaton potential, without having to first hard code functions for both the potential itself, and its first and second derivatives. This motivates us to use a symbolic interpreter. For this we use SymPy. This takes in a potential in a symbolic typed form, converts it to a python function in memory, and then similarly calculates the derivative of each. These can then be passed into the numerical solver using the equations above.

An example run for the potential $V(\phi) = m\phi^2$ would look like:

```
model = InflatonPotential("m * phi^2")

m_val = 1e-5
phi_0 = 2*np.sqrt(60)
A_s = ((1/(24*np.pi**2))*((1/(2*50+1))**(-1))*(m_val*(4*50 + 2)))

model.initial_conditions(phi_0=phi_0, a_0=1.0, m_val=m_val, k_mode=0.05,
↪  M_Pl=1.0)

Pk = model.estimate_power_spectrum(k_min=50, k_max=1e5, k_num=10, phi_0=phi_0,
↪  a_0=1.0, m_val=m_val, M_Pl=1.0, get_func=False, plot=True)
```

The outputs to this will be presented in the next section. The choice of the values for the parameters `phi_0` and `A_s` will also be discussed.

The current state of the code requires some constant m to be somehow included in the potential. This is a relic of the order in which I developed parts of this code and I hope to remove it in due course.

1.1 Example Inflaton Potentials

We will present here the analytical treatment of a few 'baby-toy' examples of inflaton potentials, obtaining the relevant values of the PPS parameters and then present the comparison with the computed PPS from the code we developed. We will mainly focus on the comparison of the spectral scalar index n_s[3]. For now we use m as a proportionality constant to keep the models compatible with the current state of the code. We will also present the tensor-scalar ratio and compare it to observed values from the literature.

1.1.1 $V(\phi) \propto \phi^2$

Let $V(\phi) = m\phi^2$, $m \in \mathbb{R}$. We define the potential slow-roll parameters as

$$\varepsilon_v = \frac{M_{\rm Pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2, \qquad \eta_v = M_{\rm Pl}^2\frac{V_{,\phi\phi}}{V} \tag{1.10}$$

Slow-roll inflation occurs when these are much less than 1.

Setting $M_{Pl} = 1$, and using the specified form of V we get

$$\varepsilon_v = \frac{1}{2}\left(\frac{2m\phi}{m\phi^2}\right)^2 = \frac{2}{\phi^2} \tag{1.11}$$

$$\eta_v = \frac{2m}{m\phi^2} = \frac{2}{\phi^2}. \tag{1.12}$$

Motivated by aiming for a slow-roll approximation we impose that $\varepsilon_v = \eta_v \ll 1$ which gives the condition $\phi \gg \sqrt{2}$. We take the value of ϕ at the end of inflation to be $\sqrt{2}$.

We have the equation

$$N_{\rm tot} \approx \int_{\phi_i}^{\phi_{\rm e}} \frac{1}{\sqrt{2\epsilon_v}}\frac{|{\rm d}\phi|}{M_{Pl}} \tag{1.13}$$

which tracks the number of e-folds inflation lasts (at least, in the local universe). To solve the flatness problem we expect we need $\gtrsim 60$ e-folds.

$$\begin{aligned} N_{\rm tot} &\approx \int_{\sqrt{2}}^{\phi_i} \frac{1}{\sqrt{4/\phi^2}}\,d\phi = \int_{\sqrt{2}}^{\phi_i}\frac{\phi}{2}\,d\phi \\ &= \left[\frac{\phi^2}{4}\right]_{\sqrt{2}}^{\phi_i} = \frac{\phi_i^2}{4} - \frac{1}{2} \end{aligned} \tag{1.14}$$

so to obtain a great enough number of e-folds we get the requirement that the initial condition on $\phi_i \gtrsim 2\sqrt{60}$. This gives the initial condition we used in the code snippet earlier.

Similarly, the number of e-folds between a specific time and the end of inflation can be given by

$$N_* \approx \int_{\phi_{\rm end}}^{\phi_*} \frac{1}{\sqrt{2\epsilon_v}}\frac{|d\phi|}{M_{\rm Pl}} = \frac{\phi_*^2}{4} - \frac{1}{2} \tag{1.15}$$

[3] Admittedly, this choice is rather motivated by the current problems with the code that we present later.

and we get a form here $N_* = \sqrt{4N_* + 2}$. Rewriting the potential slow-roll parameters in terms of N_*, we get

$$\varepsilon_{v,*} = \frac{1}{2N_* + 1} = \eta_{v,*}. \tag{1.16}$$

We also have the standard slow-roll approximation expressions, in terms of the potential slow-roll parameters, for the PPS amplitude A_s and spectral scalar index n_s:

$$A_s = \frac{1}{24\pi^2}\frac{1}{\varepsilon_{v,*}}\frac{V_*}{M_{pl}^4} \tag{1.17}$$

$$n_s - 1 = -6\varepsilon_{v,*} + 2\eta_{v,*} \tag{1.18}$$

which similarly can be expressed in terms of N_* (for the specific inflaton potential) like

$$A_s = \frac{1}{24\pi^2}\left(\frac{1}{2N_* + 1}\right)^{-1} m(4N_* + 2) = \frac{m}{12\pi^2}(2N_* + 1)^2 \tag{1.19}$$

$$n_s - 1 = -6\left(\frac{1}{2N_* + 1}\right) + 2\left(\frac{1}{2N_* + 1}\right) = -\frac{4}{2N_* + 1} \tag{1.20}$$

Let us work on the scales of the CMB, and thus we set $N_* = 50$. From this we obtain

$$n_s = 1 - 4\left(\frac{1}{101}\right) \approx 0.96$$

. This is very close to being within the allowed values for n_s from Planck. This model is discounted, however, since it predicts a value of the tensor-to-scalar ratio that is far from agreement with modern bounds.

$$r = 16\varepsilon_{v,*} = \frac{16}{2N_* + 1} \tag{1.21}$$

which, when $N_* = 50$ gives $r \approx 0.158$ which is far higher than the constraint $r < 0.035$ from the BICEP/Keck collaboration and thus the model is ruled out.

The current state of the code produces the following output,

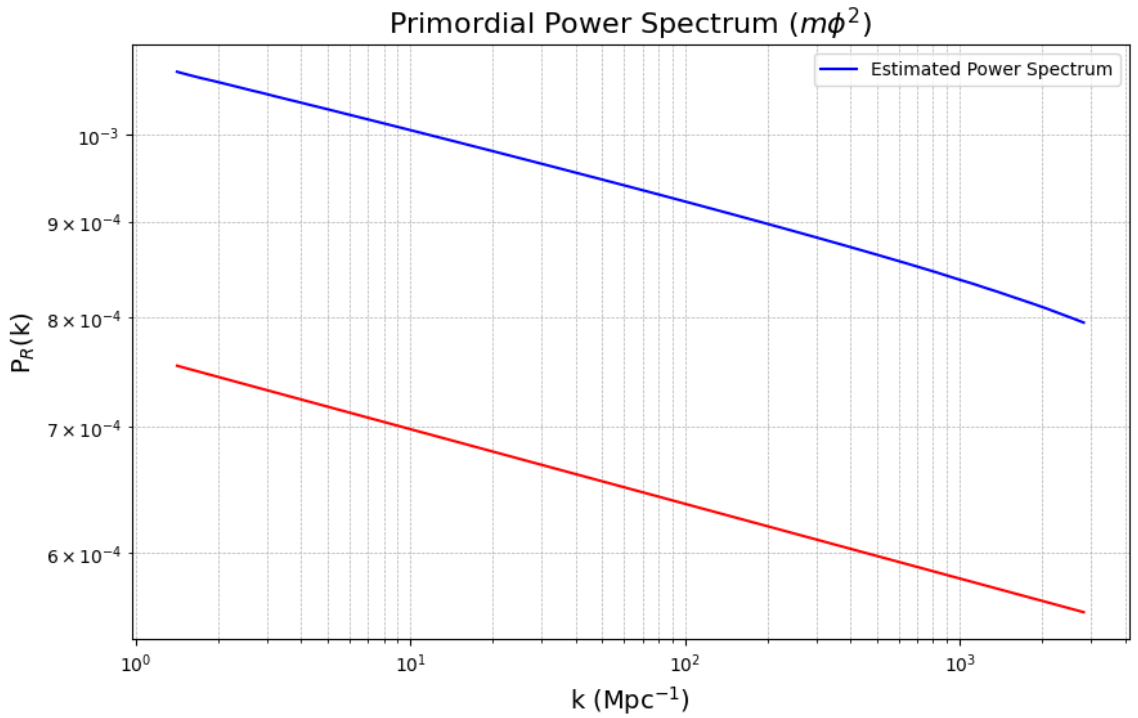


Figure 1: The current state of the code produces the following output. The blue line is the PPS approximated from successive calculations of the value at a discrete set of values for k and then fitting a functional form to these points. The red line represents the standard power law PPS that we would expect to recover in slow-roll potentials. There is clearly a disparity that needs to be worked on.

We see that this is not quite accurate. The red line is the standard power law using the values calculated above, with A_s given by 1.19 where we set $m = 1 \times 10^{-5}$. Although we recover a promising match in the gradient (n_s) we do not have a matching amplitude. We will continue to try to fix this over the coming weeks. The gradient, though more promising, is admittedly not perfect; this suggests using running of the spectral scalar index, since there is clearly some scale dependence.

1.1.2 $V(\phi) \propto \phi^4$

Let $V(\phi) = m\phi^4$.

Then, proceeding as before, we have

$$\epsilon_v = \frac{8}{\phi^2} \tag{1.22}$$

$$\eta_v = \frac{12}{\phi^2} \tag{1.23}$$

Taking the more restrictive value we set $\phi_{\text{end}} \coloneqq 2\sqrt{3}$. This gives us a necessary initial value for the field $\phi_i > \sqrt{8(60 + 3/2)}$.

Continuing in this manner we have

$$n_s = 1 - 6\left(\frac{8}{8N_* + \frac{3}{2}}\right) + 2\left(\frac{12}{8N_* + \frac{3}{2}}\right) \tag{1.24}$$

and

$$r = \frac{16 \times 8}{8N_* + \frac{3}{2}}. \tag{1.25}$$

These evaluate to, when $N_* = 50$, the following values

$$n_s \approx 0.940 \tag{1.26}$$
$$r \approx 0.319 \tag{1.27}$$

which both disagree with the current constraints, with the r-disagreement becoming significantly more severe.

2 Next Steps

I will present a quick to do list that I will aim to get done over the next week:

- Write up a full explanation of the shape we see in the matter power Spectrum
- Continue to iron out bugs in the code
  - Add scalar index running
  - Add more descriptive comparisons between theory and results
  - Try to reduce the amplitude discrepancy.
- Start a literature review for more interesting/exotic power spectra and why they are theorised (e.g. PBHs, JWST high red-shift galaxies)

I expect the coming week to be host a low level of productivity, as I am taking a trip to Sheffield as well as having a Differential Geometry assignment due. I will, of course, continue to dedicate as much time as possible to this project.